論文

An Analysis of Figural Images by Invariants of the Graph Theory*

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Abstract

This study is a pilot survey of cognitive processes by the use of invariants of the theory of graphs. Six-point, six- or five-line figures taken from a CRT display were transformed by subjects in a problem solving procedure. Each figure was changed to an indicator format (including invariants and realizing value indicators). Nonrandomness of the subjects' data represented by the indicators was tested by the chi-square goodness of fit.

The following inferences were drawn from the results: (a) subjects' figural images were rather holistic but not isomorphic; (b) many invariants were kept constant while a few realizing value indicators were actively being changed; (c) the figures produced generally took good Gestalt; and (d) subjects refrained from easily recognizable figures redundantly.

* This is a minor revision of the original article first prepared in 1983 but has not been published. A short report based on the article was mede on the occasion of the 49th JPA annual convention. Broad reexamination on the set of indicators treated here has been conducted and ensuing modification of the computation programs is now under way.

This study is a pilot survey of the cognitive processes in the solution of figure transformation problems by the use of invariants of the theory of graphs. By adopting this approach, successive figure transformations by the subjects can be sufficiently represented by a set of numerals, thus enabling a quantitative investigation of these processes. The primary purpose of the study is to discover regularities of any kind from the sequences of numbers obtained.

The task requested of subjects was to transform an initial figure into a goal figure by selecting a number of legitimate manipulations. Though this study made experimental use of a problem solving procedure, its implications are not necessarily restricted by the individual problem solutions. Because a fairly large number of randomly generated problems (i.e., pairs of initial and goal figures) were presented to the subjects, any restrictive effects caused by the specific nature of each problem were assumed to be randomized. Based on this assumtion, the general nature of the strategies found in the cognitive task and the distinctive features (i.e., figural images) are discussed in this paper.

One of the most favorable aspects of employing a problem solving technique is that it allows the subject rather extensive freedom of choice of overt actions. In other words, in order for a study of cognitive processes such as problem solving to be objectively describable, internal processes must as much as possible be externalized. This study has several merits in this respect: (a) as the task is rather easy, the internal processes of the subject are directly reflected in overt actions (the selection of figure manipulations) in small steps without much mental rehearsal; (b) the subject is not interfered with by being asked to make verbal reports, as required in many other experments, although acquiring the knack of "hitting" a desired point with a light pen might have an interfering effect at the beginning; and (c) the presented stimuli are well controlled and the outcome of the subject's manipulations unambiguous.

There are two possible levels of investigation of the figural images. The first level (level 1) is to select, rather arbitrarily, several aspects of the stimulus figures and to discuss the cognitive mechanisms of the subjects in terms of these aspects. Examples of work done at this level of investigation include that by Hochberg and McAlister (1953), who discuss the apparent bi- and tridimensionality of the Kopfermann "cube" in conjunction with its stimulus characteristics (line segments, angles, points of intersection); and that by Attneave (1957), who proposes an equation that says that the perceived complexity of a random polygon is determined by its turns, symmetry, and angular variability. Level 2 investigations are based on the assumption that a certain subset of figures taken from all the figures constitutes the predominant images. One such investigation is that of Garner and Clement (1963), who introduce the concept of "equivalence set". However, if an appropriate set of indicators (as in level 1 investigations) can group all the figures into prescribed sets (level 2), the two levels can be integrated. The present study satisfies the condition for integration with respect to isomorphism (i.e., the prescribed sets).

The concept of graph [1] and graph invariants can be used to describe

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[1] According to the definitions by Harary (1969), a graph consists of a set V of p points together with a prescribed set X of q pairs of distinct points in V. A graph with p points and q lines is called a (p, q) graph. Two graphs G and H are isomorphic if there exists a one—to—one correspondence between their point sets which preserves adjacency. An invariant of a graph G is a number associated with G which has the same value for any graph isomorphic to G.

the kinds of figures used in the experiment. If an appropriate set of invariants is selected, all isomorphic groups of figures are distinguishable. The set of indicators (invariants and realizing value indicators) employed in this study can describe large varieties of stimulus characteristics of respective figures. The term "realizing value" is used here to mean a value representing the number or location of line crossings, or the direction of figural arrangement, or a value associated to an invariant that specifies a certain configuration out of many isomorphic graphs.

If the subject's data as represented by an indicator depart significantly from randomness, they are regarded as a manifestation of cognitive activities. But the indicators are not mutually independent and there is no complete knowledge of the relationships among invariants in the theory of graphs. The concept of randomness must therefore be derived from procedural randomness. A computer simulation of the solution procedure in which random manipulations were performed was conducted to compute the marginal probability distribution for each indicator.

The main assumption for nonrandomness is that, if the subjects somewhat focus their attention on certain features of the figures ("feature" is used here to mean an aspect of a figural configuration corresponding to a specific indicator), the resulting distributions of the corresponding indicators will differ from distributions obtained by random manipulations. In analyzing the respective indicators, two possible ways of focusing by the subject will be detected in the solution process: (a) focusing on the feature of figures to be fixed, and (b) focusing on the feature of figures to be changed. These two types of focusing are categorized through the deviation in probability between the distribution obtained by the subjects and the expected (random) distribution.

The most distinctive aspect of this approach can be summarized as the comprehensive description of rather limited cognitive activites, although the experimental design and methods of analysis employed in the present study are not yet fully established.

The figures used in this experiment are constructed with six points and five or six lines. This type of figure is employed because, if the positions of the six points are fixed (like those of the regular hexagonal vertices in this experiment), the number of distinct figures possible with five lines is 3003 and the number possible with six lines is 5005, either of which is too large for a subject to memorize all the figures.

Method

Subject

Six graduate students of psychology, including both males and females.

Apparatus

A PDP 11/34 computer linked with a VT 11 display screen and a light pen.

Procedure

The subjects were asked to change the shape of a figure from a given initial state to a goal state, which transformation comprised one problem. They were able to update a figure by a prescribed sequence of light pen hittings on the light pen sensitive elements on the screen, a process referred to here as manipulation. Since the subjects had no previous experience with this kind of task, they were given one or two practice problems in order to become adept at manipulating the figures. Each subject usually solved ten problems in one session, and most of them underwent six sessions.

Experimental design

Each problem solving task was made up of a series of pictures, with a picture serving as the unit of the problem solving process. There were two main kinds of pictures: goal pictures and manipulative pictures.

1. *Representation of figures.* Throughout the experimental program, the format of figural representation consisted of line definition. That is, any figure used in the experiment can be represented by a set of specific lines, which are defined as N pairs of uniquely labeled points. Here, N is the number of lines making up a figure. Only when a picture was produced on the CRT screen was the line definition format given a specific realization. The points used as the ends of the lines forming the figures were positioned at the vertices of a regular hexagon with sides 4.66 cm long. The point on the upper right was labeled (1) and the other points labeled (2) to (6) counterclockwise from point (1).

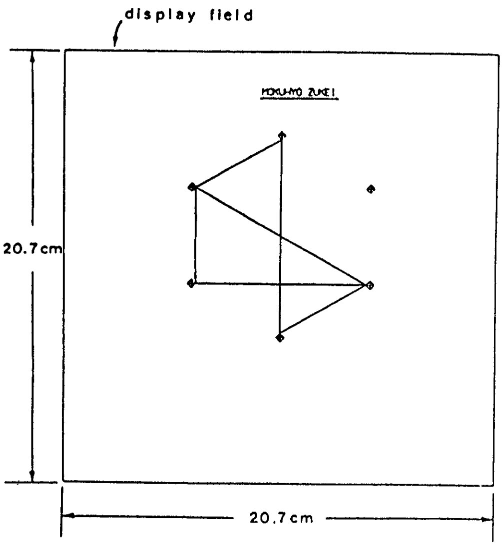

Figure 1. An example of a goal picture (MOKUHYO ZUKEI : goal figure).

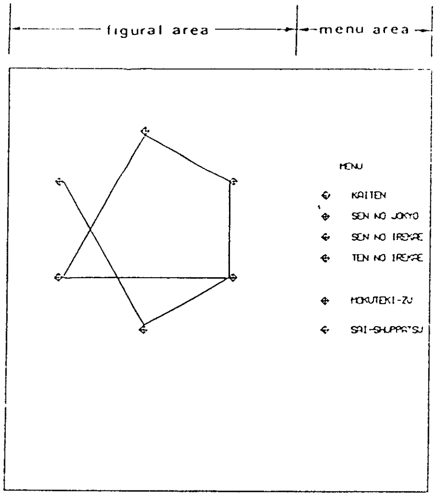

Figure 2. An example of a manipulative picture (KAITEN : rotation, SEN NO JOKYO : remove a line, SEN NO IREKAE : switch lines, TEN NO IREKAE: switch points, MOKUTEKI—ZU: goal figure, SAI—SHUPPATSU: restart).

2. *Goal picture.* Figure 1 shows an example of a goal picture, containing the figure the subject should arrive at. This picture was displayed at the start of a problem and when the subject chose the goal reference manipulation. The size of the figures in the goal pictures did not differ from those in the manipulative pictures, but the latter were displaced 3.44 cm leftward and 1.01 cm upward relative to the former.

3. *Manipulative picture.* Figure 2 is an example of a manipulative picture. The subject's light pen hittings were effective only on this type of picture. The manipulative picture was divided into three areas : a menu area, a figural area, and a message string area. A manipulation was set off when one of the points (a point was symbolized as a cross enclosed in a rhombus) in the menu area was touched by the

light pen. If this was either "rotation (KAITEN)", "goal reference (MO-KUTEKI—ZU)", or "restart (SAI—SHUPPATSU)", the picture was immediately updated.

The "rotation" manipulation rotated the original figure 60° counter-clockwise, but left the shape of the figure unchanged.

Selection of the "goal reference" manipulation cleared away the current picture and temporarily dispalyed the goal picture. This was followed by brief masking, after which the original picture (on which the "goal reference" manipulation had been selected) was restored.

The "restart" manipulation merely replaced the picture with the initial figure (the initial state); no further change of the picture was introduced until the subject made another manipulation.

If a subject wanted to perform a manipulation other than one of those mentioned above, he/she had to select the lines and/or symbolized points (i.e., the light pen sensitive elements) in the figural area necessary to complete the manipulation.

Selection of the "remove a line (SEN NO JOKYO)" manipulation remove a specified line from the figure. This was done by first hitting the point for this item in the menu, then hitting the line of the figure in the figural area that the subject wished to delete. When the number of lines in the goal figure was equal to that in the current figure, selection of this manipulation was blocked.

The "switch lines (SEN NO IREKAE)" manipulation was completed by first hitting (1) the appropriate point in the menu area, then (2) any line in the figure, and finally (3) two points not yet joined by a line. The line touched in step (2) was deleted from the existing figure and the newly defined line between the two points specified in step (3) added.

The steps followed for the "switch points (TEN NO IREKAE)" manipulation were (1) hitting the appropriate point in the menu, then (2) consecutively hitting two non-identical points in the figure. The lines incident to the first point were transposed to the second point and, conversely, the lines incident to the second point were transposed to the first point.

If the subject did not make any light pen hittings on the manipulative picture within 7 seconds, the screen was cleared away for 1.5 sec. This was intended to prevent mental rehearsal by the subjects and to externalize their cognitive processes as much as possible.

4. *Presentation of pictures.* When the subject touched the ready sign (a stylized point in a blank screen) at the start, the system displayed the goal picture for a duration of 2.4 sec. This was followed immediately by masking (0.3 sec), which in turn gave place to the message informing the subject of the start of the problem (4 sec). The manipulative picture (initial state), then appeared. If, in the course of solving a problem, the "goal reference" manipulation was selected, the goal picture was presented for a duration of 1.2 sec, and the ensuing masking for 0.3 sec.

For more detailed explanation and illustrations of the experimental design, see Kanbe (1983).

Problem generation

Before presenting a problem (consisting of an initial figure and a goal figure) to a subject, both figures were randomly produced by the random assignment of point labels in the line definition format. Two kinds of problems were generated: (a) 6—6 type problems, in which both the initial and the goal figure consisted of six lines; and (b) 6—5 type problems, in which the initial figure consisted of six lines, and the goal figure five lines.

All the problems to be generated for a given subject were recorded and the same problem was never presented twice.

Simulations of random manipulations

These computations were intended to estimate the marginal distributions along the values for each indicator by simulating completely random selection of manipulations by imaginary subjects.

Three kinds of simulations were conducted. One was of 6—6 type problems in which the simulated manipulations were restricted to only (6, 6) figures. The menu selections (excluding the "remove a line" manipulations) and the element selections were made by means of random numbers of uniform distribution. The second was of 6—5 type problems in which the menu selections other than the "remove a line" manipulation, and the selections of the elements (lines and points) were random. But within any given simulated problem solution process, whose length (number of manipulations) had been predetermined, once the number of lines of the current figure became the same as in the goal figure, selection of the "remove a line" manipulation was blocked. The third type of simulation was of 5—5 type problems in which the manipulations were restricted to (6,5) figures only. Here the stage following the "remove a line" manipulation was simulated.

The iteration limit, or total number of simulated manipulations, was 5000 for the 6—6 type, 8000 for the 6—5 type, and 5000 for the 5—5 type. In all the simulations, the length of a problem solution was determined by the normally distributed random numbers, the mean and the standard deviation of which were set at 6.0 and 2.0, respectively. The changes to the indicator format, which consited of the invariants and realizing values, were made immediately after completion of each manipulation.

Change to indicators

Each figure produced by a subject was changed from a line definition format to an indicator format. A list of the indicators employed in the study is given below with abbreviations and brief explanations for each. The terms and definitions mainly conform to Harary's (1969) [2].

1. *Number of trials* (TRIL). Each manipulative picture, which constituted one trial, was numbered sequentially from the initial state to the goal state. If a subject selected the "goal reference" manipulation, the manipulative picture just before the appearance of the goal picture was not counted as a trial.

2. *Types of manipulations* (MTYP). Specific manipulations selected by a subject. Label (1) was assigned to the "rotation" manipulation, label (2) to the "remove a line", (3) to the "switch lines", (4) to the "switch points", (5) to the "restart", and (6) to the "goal reference".

3. *Time consumed* (TIME). The time consumed in a trial was measured and recorded in units of 0.02 sec starting at the presentation of the manipulative picture on the CRT until completion of the picture.

4. *Number of lines* (LINE). The number of lines contained in a given figure.

5. *Number of cycles* (CYCL). A cycle is a closed walk (an alternating sequence of points and lines $v_0, x_1, v_1, \cdots, v_{n-1}, x_n, v_n$, begining and

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[2] Definitions derived by the author (some with slight modifications) are included in the explanations of CYCL, CCMF, PCOV, NCRI, RADS, NCET, CMPT, NCUT, MXDG, NISL and NEND.

ending with points, where $v_0 = v_n$), if $n\,(\geqq 3)$ points are distinct.

6. *Circumference* (CCMF). This is the length of the longest cycle(s) in a given graph G.

7. *Point covering number* (PCOV). This is the smallest number of points in any point cover for G (a point and a line are said to cover each other if they are incident, and a point cover is a set of points which covers all the lines of a graph G).

8. *Number of critical points* (NCRI). If $\alpha_0\,(G - v) < \alpha_0\,(G)$, then v is called a critical point, whilst α_0 is a point covering number and subgraph $G - v$ results from removal of a point v from a graph G.

9. *Location of critical points* (LCRI). On LCRI and hereafter, the location of the point(s) concerned was defined as the geometrical [3] centroid of this(ese) point(s). The location was expressed by the regional representation, which will be described later.

10. *Radius* (RADS). The eccentricity of a point v in a connected graph G is the maximal distance from v, and the radius is defined as the minimum eccentricity of the points. Here, the distance is the length or the shortest walk in which all points are distinct.

11. *Number of central points* (NCET). If the eccentricity of a point is equal to the radius of G, the point is called a central point.

12. *Location of central points* (LCET)

13. *Number of components* (CMPT). A component is a maximally connected subgraph of G. An isolated point (explained below) also is considered as a component.

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[3] As this indicator is a realizing value and, thus, a concept not within the graph theory, the adjective "geometrical" was added in order to distinguish it from the same term used in the graph theory.

14. *Number of cutpoints* (NCUT). A cutpoint in G is a point whose removal increases the number of components.

15. *Location of cutpoints* (LCUT)

16. *Maximum degree* (MXDG). The degree of a point is the number of lines incident with that point.

17. *Number of maximum degree points* (NXDG)

18. *Location of maximum degree points* (LXDG)

19. *Number of isolated points* (NISL). A point whose degree is equal to 0 (i.e., not adjacent to any other point) is called an isolated point.

20. *Location of isolated points* (LISL)

21. *Number of endpoints* (NEND). An endpoint is a point whose degree is equal to 1.

22. *Location of endpoints* (LEND)

23. *Geometrical centroid of lines* (ROID). The geometrical centroid of all the line segments, which together constitute a current figure, was computed and indicated by the regional representation.

24. *Number of geometrical crossings of lines* (CRSS). This is the number of crossings of lines in a current figure for a given arrangement of points (positioned at the vertices of the regular hexagon).

25. *Location of geometrical crossings of lines* (LCRS)

26. *Direction among plural components* (DIRC). If there existed two or more components in a current figure, a slope of linear regression among the ROID's of each component was computed and expressed by the slope representation. This representation was obtained by converting the coefficient of the linear regression b to the six labels (0), (1), (2), (3), (4), (99) according to Table 1.

Table 1

Slope Representation

Slope	Label
$-.421 < b < .421$	(0)
$.421 \leqq b \leqq 2.375$	(1)
$2.375 < b$ or $b < -2.375$	(2)
$-2.375 \leqq b \leqq -.421$	(3)

Note. b is the coefficient of linear regression. If two centroids of the components fell on the same point, the label (4) was assigned. If there was only one component, the label (99) was assigned.

Regional representation. As stated earlier, the indicators dealing with the location of a point followed the regional representation. The plane in which the figures were contained was divided into seven semiclosed regions labeled (1) to (7). Any point in the plane was represented by the label of the region which contained the point (see Figure 3). If there existed no such point, label (99) was assigned.

The above listed indicators might be grouped into two kinds. TRIL, MTYP, TIME can be called external indicators in the sense that they deal with certain modes of the subject's responses and not with the figures per se. The others are internal indicators because they deal with various aspects of the figures. The use of such internal indicators is advoacted for a much closer analysis of the cognitive processes

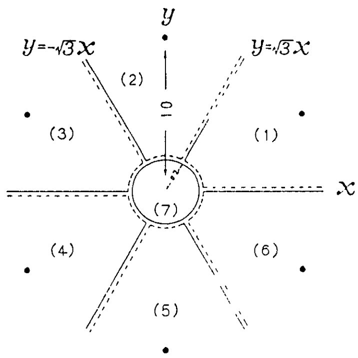

Figure 3. Regional representation 〔The regular hexagonal field on the CRT screen was divided into seven closed subregions labeled (1) to (7). The radius of the central region (7) is one fifth of the radius of the circumcircle of the six points. The inclusion of a boundary between the two contiguous regions is indicated by a solid line.〕

themselves. Of the internal indicators, LINE, CYCL, CCMF, PCOV, NCRI, RADS, NCET, CMPT, NCUT, MXDG, NXDS, NISL, and NEND are invariants. Other internal indicators (realizing value indicators) give information on any specific configuration within identical isomorphic graphs (within which the values of the invariants are identical) for the given arrangement of six points.

Results and Discussion

The daha on the subjects were classified according to the two major types of problems, 6—6 type problems and 6—5 type problems. The ensuing treatment of these data were thereafter conducted discriminately. An additional problem type, the 5—5 type, was later included in a detailed analysis.

External indicators

The number of 6—6 type problems presented to the subjects was 156 and the total number of figures (i.e., trials) produced by the subject's manipulations (excepting "goal reference") was 669. The mean and the standard deviation in the number of trials for a problem were 4.78 and 1.86, respectively. The mean and the standard deviation in TIME were 10.54 and 3.97 sec, respectively.

In the 6—5 type problems, the number of problems was 142 and the total number of trials, 645. The mean and standard deviation for one problem were 4.96 trials and 1.81 trials, respectively. In the course of the average solution, selection of the "remove a line" manipulation occurred around the 4.43th trial, with a standard deviation of 1.78 trials. The mean and the standard deviation in TIME were 9.50 and 3.83 sec, respectively.

The difference in TIME between the 6—6 type and the 6—5 type problems was significant: $t\ (1312) = 4.82$, $p < .001$. If we assume that TIME measures the ease of a problem (or the recognizability of the figure), the 6—5 type problems are, in general, easier than the 6—6 type problems.

Table 2 shows the probabilities of each type of manipulation in the menu being selected. In order to compare the tendencies to select the various types of manipulations, a chi-square test of the goodness of fit was conducted. The two probability distributions applied were the original distribution of the 6—6 type problems as against the corrected distribution obtained by excluding the "remove a line" from the selection repertoire in the 6—5 type problems and redistributing each manipulation to an expected probability. The result of the test indicates that there is no significant difference between the two types: $\chi^2(2)=.561$, $p > .05$.

Table 2

Probabilities for the Selection of Each Type of Manipulations In the 6—6 Type and 6—5 Type Problems

Menu:	Rotation	Remove a line	Switch lines	Switch points	Restart	Goal reference [a]
6—6 type	.115	0.0	.864	.018	.003	.380
6—5 type	.093	.219	.676	.009	.003	.333

[a] The number of "goal reference" manipulations that occurred was divided by the total number of trials ("goal reference" was not included).

The difference between the two types of problems in the probabilities of the "goal reference" being selected is also not significant (critical ratio: 1.78, $p > .05$).

These results indicate that, although the structures of the (6, 6) graphs differ widely from those of the (6, 5) graphs at the level of the indicators, the subjects' selection strategies at manipulation level do not differ in the two types of problems.

One outstanding feature in Table 2 is the dominant selection of the "switch lines" manipulation, while the selection of the "switch points" manipulation is particularly infrequent. The latter manipulation always preserves the isomorphism of the graphs, but the former does not necessarily do so. This immediately indicates that the subjects did not make manipulations while conscious of the isomorphism of the graphs. However, the "rotation" manipulation, which preserves not only the isomorphism but also the actual form of a graph, is highly preferred to the "switch points" manipulation. Consequently, it might be suggested that the subjects perceived a figure as a whole, but they were hardly able to hold an isomorphic image of a figure. The above inference coincides with reports by some subjects that the outcomes of the "switch points" were rather unpredictable from pre-manipulation impressions.

Internal indicators

Whether the subjects focused their manipulation efforts on specific features of the figures was tested by the chi-square goodness of fit on respective indicators.

As one figure assigns each indicator one integer value, a collection of many figures gives a specific frequency distribition for the values (classes) of an indicator. For each indicator, the frequency distribution obtained by the subjects' actual manipulations was compared with the distribution obtained from randomly simulated manipulations. When the frequency of one class (value) of an actual distribution was less than 5, this was united with an(other) class(es) to form a new class with a frequency of 5 or more. Indicators resulting in significant difference on a 5 % level or above by this test will henceforth be called "picked-up" indicators.

Moreover, if a signifcant difference is due to the fact that the absolute deviation of the observed from the expected probabilities is predominantly large in a certain class (value) of an indicator, the state of the feature may constitute a determinant of the subjects' images. A criterion was therefore tentatively set up to classify indicators into three categories: (a) an indicator whose positive deviation in a specific class is somewhat higher than the averaged absolute deviation of the other classes (positive fixed indicators abbreviated as F+), (b) an indicator whose absolute value of the negative deviation in a specific class is somewhat higher than the averaged absolute deviations of the other classes (negative fixed indicators, abbreviated as F−), and (c) indicators which do not belong to either the F+ or F− categories (abbreviated as V).

The criteria [4] were set up so that the absolute deviation of one class is 25 % larger than the averaged absolute deviations of all the other classes. And if the absolute deviation of one class exceeds a criterion, the class is called a peak.

The primary assumption of this categorization is that the subjects will (F+) or will not (F−) produce figures that possess certain states of features (i.e., certain classes of the corresponding indicators in the indicator format) in common more frequently than mere chance in the course of solutions. The subjects are thought to focus the features on certain fixed states in the case of F+ indicators, and to avoid producing figures containing certain states of the features in the case of F− indicators.

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[4] Thus, the relation between number of classes (N) and criterion (P: expressed by the ratio of the absolute deviation of one class to the total absolute deviations) becomes as follows.

If $N=3$, then $P=.5$;
$N=4$, $P=.438$;
$N=5$, $P=.4$;
$N=6$, $P=.375$;
$N=7$, $P=.357$.

6 — 6 type problems

Table 3 shows the picked-up indicators and their attributes.

Table 3

Picked-up Indicators in the 6—6 Type Problems

Indicator	*df*	χ^2	Category	Peak	Range [a]
CYCL	3	29.11**	F−	1	1,2,3,7
			F+	3	1,2,3,7
CCMF	3	21.05**	F−	3	3,4,5,6
PCOV	1	8.60**	V		2,3,4
NCET	5	35.07**	F+	5	1,2,3,4,5,6
CMPT	2	26.34**	F−	1	1,2,3
NCUT	3	17.71**	F+	0	0,1,2,3
LCUT	7	25.91**	V		1,2,3,4,5,6,7,99
MXDG	3	25.66**	F+	3	2,3,4,5
			F−	4	2,3,4,5
NXDG	4	18.44**	F−	1	1,2,3,4,6
NISL	2	26.03**	F−	0	0,1,2
LISL	6	36.55**	F−	99	1,2,3,4,5,6,7,99
NEND	3	24.96**	V		0,1,2,3
LEND	7	17.51*	F+	99	1,2,3,4,5,6,7,99
LCRS	7	18.98**	V		1,2,3,4,5,6,7,99
DIRC	4	128.30**	V		0,1,2,3,4,99

[a] All the values each indicator theoretically takes.

* $p < .05$.

** $p < .01$.

According to the table, the picked-up indicators belonging to the F+ and F— categories are mostly invariants, while those belonging to V are largely realizing value indicators. Hence, the first conjecture derived from the table is that the subjects proceeded with their tasks in a manner whereby they kept the abstract images of the figures (invariants) unchanged while constantly changing the superficial features (realizing value indicators). Furthermore, most of the invariants in F— take the minimum value of each range and the invariants in F+ the mediant value of each range. The exception were NCUT (0) [5] in F+ and MXDG (4) in F—.

It must be noted that value 0 of an N—indicator (an invariant which indicates the number of points concerned) is identical with value 99 of a conceptually correlating L-indicator (a realizing value indicator which gives locational information). And this is the case with NISL (0) and LISL (99) in category F—.

Interpreting the indicators picked up in F+ in more familiar terms, the subject's dominant fixed images seem to be: (a) a relatively large contour containing two smaller closures 〔CYCL (3)〕, (b) symmetrical 〔NCET (5)〕 structures, (c) compact 〔NCUT (0)〕 structures, and (d) a most frequently occurring (i.e., most indistinctive) type of line concentration 〔MXDG (3)〕. The common aspect of the figures assuming these indicators' state is the closedness (i.e., no point is an endpoint) of their shapes. In short, these features could be summarized as popular (or indistinctive), but good Gestalt.

When an invariant takes a minimum value, the corresponding feature

[5] The parenthesized digit indicates the peak value of the picked-up indicator.

is usually well separated from other features. When the value of MXDG is 4, for example, the only value of NXDG is 1. Thus MXDG (4), the only non-minimum picked-up indicator in F−, should be regarded in conjunction with NXDG (1) in F−, and not treated separately. The values of the indicators in F− thus seem to suggest that the subjects avoided producing well-separated (or highly recognizable) figures with respect to the features concerned.

Both LCRS and DIRC are indicators that are dependent upon the kind of figure formed by the specific arrangement of the six points and are therefore not derived from the graph theory. The subjects are considered to have focused their efforts on manipulating the location of line crossing(s) and the main axis (DIRC) while intentionally or unintentionally trying to change these two features. If so, the figural images (or strategies) of the subjects are not of a static nature as far as the superficial cues (LCRS, DIRC) are concerned.

The chi-square value of DIRC is especially large. As DIRC gives information on a broader area rather than a particular point, the subjects are conjectured to have applied some kind of holistic approach in their problem solutions.

But there is another possibility. In F−, CMPT (1) and NISL (0) are picked up. These two indicators take only three values: 1, 2, 3 for CMPT, and 0, 1, 2 for NISL. Moreover, the expected probabilities of CMPT assuming value 3 and NISL assuming value 2 are both fairly small, although greater than 0. Thus, the tendencies of CMPT (1) and NISL (0) belonging to F− are of roughly equal significance to those of CMPT (2) and NISL (1) belonging to F+, although to a slightly lesser degree. If CMPT (2) occurs at a frequency greater than mere chance, DIRC is also highly likely to be picked up in some category. If we take this interpretation, what the subjects focused on was not superficial cues in the figures (DIRC) but deeper structural features (CMPT or NISL). At present, which of these interpretations is most appropriate cannot be determined.

6 — 5 type problems

Comparing Table 4 with Table 3, the number of indicators picked up is less in the 6—5 type problems than in the 6—6 type problems. This may mean that the analysis fails to fully grasp the cognitive processes of the subjects.

Table 4

Picked-up Indicators in the 6—5 Type Problems

Indicator	*df*	χ^2	Category	Peak	Range
LINE	1	47.78 **	V		5,6
CYCL	3	19.04 **	F−	0	0,1,2,3,7
			F+	3	0,1,2,3,7
CCMF	4	30.27 **	V		0,3,4,5,6
PCOV	1	10.10 **	V		1,2,3,4
NCRI	4	18.43 **	F+	6	1,2,3,4,5,6
NCET	5	14.42 *	V		1,2,3,4,5,6
LCET	6	13.85 *	V		1,2,3,4,5,6,7,99
NXDG	4	25.05 **	F−	1	1,2,3,4,5,6
LXDG	6	18.12 **	V		1,2,3,4,5,6,7,99
DIRC	5	250141.27 **	F−	0	0,1,2,3,4,99

* $p < .05$.

** $p < .01$.

Most of the picked-up indicators in the 6—5 type problems are included in the picked-up indicators in the 6—6 type problems, irrespective of their categories. These are CYCL, CCMF, PCOV, NCET, NXDG and DIRC.

Both the (6, 6) and (6, 5) figures are covered in the analysis of the 6—5 problems. There is thus a possibility that the dominant cognitive processes occurring in the (6, 6) figure stage overwhelm those in the (6, 5) stage. At manipulation level, this means that the subjects chose the "remove a line" manipulation, which results in (6, 5) figures, only after they felt confident of the images of the goal figures (or sensed that the goal was at hand) at the final stage of the solutions (see *External indicators*). That is, the subject's efforts at focusing were primarily made during manipulations on (6, 6) figures; by the time they had chosen the "remove a line" manipulation, their efforts were almost at an end.

Table 5

Picked-up Indicators in the (6,5) Figure Stage

Indicator	*df*	χ^2	Category	Peak	Range
LCET	6	12.73 *	V		1,2,3,4,5,6,7,99
DIRC	4	90.25 **	F—	0	0,1,2,3,4,99

* $p < .05$.

** $p < .01$.

This interpretation is further supported by Table 5, which shows that only two indicators are picked up in the goodness of fit test applied to the subjects' (6, 5) figures. The two realizing value indicators picked up, LCET and DIRC, suggest that only a minor adjustment in the superficial features is made at the (6, 5) figure stage.

If the above statement (the predominance of (6, 6) figure processes) is supportable, it would be natural to assume that the characteristics discussed in the 6—6 type analysis are also revealed in this analysis. In general, the results vindicate this assumption. (a) The indicators in categories F+ and F— are mostly invariants. (b) The peaks of the invariants in F— take the minimum values of the ranges. (c) The chi-square value of DIRC is very large. (d) Several conceptually correlating indicators are picked up in pairs (CYCL and CCMF; PCOV and CRIN). But there were a few notable differences. Namely, (e) the number of realizing value indicators belonging to category V are not predominant, and (f) DIRC belongs to F—.

With regard to point (e), the invariants in category V are CCMF, PCOV and NCET. They have already been picked up in the analysis of the 6—6 type problems. The classification of PCOV in category V is based on the same grounds (i. e., high correlated pick up with NRCI) as those already discussed in the former section.

It is probable that the tendency for CCMF and NCET to be classified as category F— and F+ respectively will be somewhat lessened by the continued performance of manipulations at the (6, 5) figure stage.

With regard to point (f), DIRC (0) is also categorized as an F— indicator by the additional analysis of (6, 5) figures (see Table 5). Thus this categorization appears quite certainly to result from the processing done on the (6, 5) figures.

Conclusions

In the preceding sections, several types of evidence concerning cognitive processes were discussed.

1. Although the structural difference is wider in the 6—5 type problems than the 6—6 type problems, the former were easier too solve. This indicates that (6, 5) figures can more easily be recognized and retained.

2. The subject's strategies in selecting manipulations did not differ between the 6—6 type and the 6—5 type problems.

3. The subjects took a holistic approach in the figure transformations, but their images were not isomorphic.

4. Abstract features (invariants) tended to be kept constant while a few superficial features (especially LCRS and DIRC) were actively changed. The subject's holistic images pointed out above were thought to remain under the dominant influences of these two features.

5. The subjects seemed to avoid producing easily recognizable figures while solving their problems. This means they probably were capable of processing the transformations in rather complex states of figural formation, while retaining rough images (i.e., inexact images) of the goal figures.

6. The subject tended to produce good Gestalt figures.

7. Because the subjects proceeded their respective solutions mainly at the (6, 6) figure stage, the same tendencies were detected with the 6—5 type problems as were pointed out in 4. and 5. above. The fact the (6, 5) figures were not favoured in the manipulations would indicate, here again, that the subjects avoided easily recongizable stage 〔(6, 5) figures〕, or that they dared not prolong unnecessarily their manipulations.

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